

SOME SUPERCONGRUENCES MODULO p^2

ZHI-HONG SUN

School of the Mathematical Sciences, Huaiyin Normal University,

Huaian, Jiangsu 223001, PR China

Email: zhihongsun@yahoo.com

Homepage: <http://www.hytc.edu.cn/xsjl/szh>

ABSTRACT. Let $p > 3$ be a prime, and let m be an integer with $p \nmid m$. In the paper we prove some supercongruences concerning

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k}, \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{432^k},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \pmod{p^2}.$$

Thus we solve some conjectures of Zhi-Wei Sun and the author.

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1. Introduction.

For positive integers a, b and n , if $n = ax^2 + by^2$ for some integers x and y , we briefly say that $n = ax^2 + by^2$. Let $p > 3$ be a prime. In 2003, Rodriguez-Villegas[RV] posed some conjectures on supercongruences modulo p^2 . Three of his conjectures are equivalent to

$$(1.1) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4A^2 - 2p \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

$$(1.2) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \begin{cases} 4c^2 - 2p \pmod{p^2} & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases}$$

$$(1.3) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{1728^k} \equiv \begin{cases} \left(\frac{p}{3}\right)(4a^2 - 2p) \pmod{p^2} & \text{if } p = a^2 + b^2 \equiv 1 \pmod{4} \text{ and } 2 \nmid a, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where $\left(\frac{a}{m}\right)$ is the Jacobi symbol. The above conjectures have been solved by Mortenson[M] and Zhi-Wei Sun[Su2].

Let \mathbb{Z} be the set of integers, and let $[x]$ be the greatest integer function. For a prime p let \mathbb{Z}_p be the set of rational numbers whose denominator is coprime to p . Recently the author's brother Zhi-Wei Sun posed many conjectures ([Su1]) involving

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k}, \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{432^k}, \\ & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \pmod{p^2}, \end{aligned}$$

where $p > 3$ is a prime and $m \in \mathbb{Z}$ with $p \nmid m$. For example, Zhi-Wei Sun conjectured ([Su1, Conjectures A8 and A9]) that for any prime $p > 3$,

$$(1.4) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-192)^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \\ L^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = L^2 + 27M^2, \end{cases}$$

$$(1.5) \quad \sum_{k=0}^{p-1} \frac{(6k)!}{(-96)^{3k} (3k)! k!^3} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = -1, \\ \left(\frac{-6}{p}\right)(x^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{p}{19}\right) = 1 \text{ and so } 4p = x^2 + 19y^2. \end{cases}$$

In [S3], the author proved (1.4) and (1.5) modulo p .

Let p be an odd prime and let x be a variable. In the paper we establish the following general congruences:

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1 - 27x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \right)^2 \pmod{p^2}, \\ & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1 - 64x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}, \\ & \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1 - 432x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \right)^2 \pmod{p^2}. \end{aligned}$$

As an application, using the work in [S2, S3] we prove many congruences modulo p^2 . For example, (1.4) is true for $p \equiv 2 \pmod{3}$ and (1.5) is true when $\left(\frac{p}{19}\right) = -1$.

2. Congruences for $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k}$ and $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \pmod{p^2}$.

Lemma 2.1. *Let m be a nonnegative integer. Then*

$$\sum_{k=0}^m \binom{2k}{k}^2 \binom{3k}{k} \binom{k}{m-k} (-27)^{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{2(m-k)}{m-k} \binom{3(m-k)}{m-k}.$$

We prove the lemma by using WZ method and Mathematica. Clearly the result is true for $m = 0, 1$. Since both sides satisfy the same recurrence relation

$$81(m+1)(3m+2)(3m+4)S(m) - 3(2m+3)(9m^2+27m+22)S(m+1) + (m+2)^3 S(m+2) = 0,$$

we see that the lemma is true. The proof certificate for the left hand side is

$$-\frac{729k^2(m+2)(m-2k)(m-2k+1)}{(m-k+1)(m-k+2)},$$

and the proof certificate for the right hand side is

$$\frac{9k^2(3m-3k+1)(3m-3k+2)(9m^2-9mk+30m-14k+24)}{(m-k+1)^2(m-k+2)^2}.$$

Theorem 2.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1-27x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1-27x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} x^k \sum_{r=0}^k \binom{k}{r} (-27x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k}^2 \binom{3k}{k} \binom{k}{m-k} (-27)^{m-k}. \end{aligned}$$

Suppose $p \leq m \leq 2p-2$ and $0 \leq k \leq p-1$. If $k > \frac{p}{2}$, then $p \mid \binom{2k}{k}$ and so $p^2 \mid \binom{2k}{k}^2$. If $k < \frac{p}{2}$, then $m-k \geq p-k > k$ and so $\binom{k}{m-k} = 0$. Thus, from the above and Lemma 2.1 we deduce

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} (x(1-27x))^k \\ &\equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{3k}{k} \binom{k}{m-k} (-27)^{m-k} \\ &= \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{2(m-k)}{m-k} \binom{3(m-k)}{m-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{m=k}^{p-1} \binom{2(m-k)}{m-k} \binom{3(m-k)}{m-k} x^{m-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{r=0}^{p-1-k} \binom{2r}{r} \binom{3r}{r} x^r \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \left(\sum_{r=0}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r - \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r \right) \\ &= \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r \pmod{p^2}. \end{aligned}$$

If $\frac{2p}{3} \leq k \leq p-1$, then $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p^2}$. If $0 \leq k \leq \frac{p}{3}$ and $p-k \leq r \leq p-1$, then $\frac{2p}{3} \leq r \leq p-1$ and so $\binom{2r}{r}\binom{3r}{r} = \frac{(3r)!}{r!^3} \equiv 0 \pmod{p^2}$. If $\frac{p}{3} < k < \frac{2p}{3}$ and $p-k \leq r \leq p-1$, then $r \geq p-k > \frac{p}{3}$, $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p}$ and $\binom{2r}{r}\binom{3r}{r} = \frac{(3r)!}{r!^3} \equiv 0 \pmod{p}$. Hence, for $0 \leq k \leq p-1$ and $p-k \leq r \leq p-1$ we have $p^2 \mid \binom{2k}{k}\binom{3k}{k}\binom{2r}{r}\binom{3r}{r}$ and so

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{3r}{r} x^r \equiv 0 \pmod{p^2}.$$

Therefore the result follows.

Corollary 2.1. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left(\frac{1 - \sqrt{1 - 108/m}}{54} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1 - \sqrt{1 - 108/m}}{54}$ in Theorem 2.1 we deduce the result.

Corollary 2.2. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{[p/3]} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv 0 \pmod{p} \quad \text{implies} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. Clearly $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p}$ for $\frac{p}{3} < k < p$. Suppose $\sum_{k=0}^{[p/3]} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv 0 \pmod{p}$. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv \sum_{k=0}^{[p/3]} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{m^k} \equiv 0 \pmod{p}.$$

Using Corollary 2.1 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \left(\frac{1 - \sqrt{1 - 108/m}}{54} \right)^k \equiv 0 \pmod{p}$$

and so the result follows from Corollary 2.1.

Theorem 2.2. *Let $p \equiv 1 \pmod{3}$ be a prime and so $p = A^2 + 3B^2$ with $A \equiv 1 \pmod{3}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k} \equiv 2A - \frac{p}{2A} \pmod{p^2}.$$

Proof. Clearly $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p}$ for $\frac{p}{3} < k < p$. Using [S1, Theorem 2.5] we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k} = \sum_{k=0}^{p-1} \frac{(3k)!}{54^k \cdot k!^3} \equiv \sum_{k=0}^{[p/3]} \frac{(3k)!}{54^k \cdot k!^3} \equiv 2A \pmod{p}.$$

Set $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} = 2A + qp$. Then

$$\left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \right)^2 = (2A + qp)^2 \equiv 4A^2 + 4Aqp \pmod{p^2}.$$

Taking $m = 108$ in Corollary 2.1 we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \right)^2 \pmod{p^2}.$$

Thus, by (1.1) and the above we have

$$4A^2 - 2p \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \right)^2 \equiv 4A^2 + 4Aqp \pmod{p^2}$$

and hence $q \equiv -\frac{1}{2A} \pmod{p}$. So the theorem is proved.

Remark 2.1 Theorem 2.2 was conjectured by the author in [S1]. When p is a prime of the form $6k + 5$, it was conjectured in [S1] that $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \equiv 0 \pmod{p^2}$. This was recently confirmed by Zhi-Wei Sun[Su3].

Theorem 2.3. *Let $p \equiv 5 \pmod{6}$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-192)^k} \equiv 0 \pmod{p^2}.$$

Proof. Since $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3} \equiv 0 \pmod{p}$ for $\frac{p}{3} < k < p$, from [S3, Theorem 4.3] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k} = \sum_{k=0}^{p-1} \frac{(3k)!}{(-216)^k \cdot k!^3} \equiv \sum_{k=0}^{[p/3]} \frac{(3k)!}{(-216)^k \cdot k!^3} \equiv 0 \pmod{p}.$$

Thus, taking $x = -\frac{1}{216}$ in Theorem 2.1 we obtain the result.

Lemma 2.2. *Let m be a nonnegative integer. Then*

$$\sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)}.$$

We prove the lemma by using WZ method and Mathematica. Clearly the result is true for $m = 0, 1$. Since both sides satisfy the same recurrence relation

$$1024(m+1)(2m+1)(2m+3)S(m) - 8(2m+3)(8m^2 + 24m + 19)S(m+1) \\ + (m+2)^3 S(m+2) = 0,$$

we see that Lemma 2.2 is true. The proof certificate for the left hand side is

$$-\frac{4096k^2(m+2)(m-2k)(m-2k+1)}{(m-k+1)(m-k+2)},$$

and the proof certificate for the right hand side is

$$\frac{16k^2(4m-4k+1)(4m-4k+3)(16m^2-16mk+55m-26k+46)}{(m-k+1)^2(m-k+2)^2}.$$

Theorem 2.4. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} x^k \sum_{r=0}^k \binom{k}{r} (-64x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k}. \end{aligned}$$

Suppose $p \leq m \leq 2p-2$ and $0 \leq k \leq p-1$. If $k > \frac{p}{2}$, then $p \mid \binom{2k}{k}$ and so $p^2 \mid \binom{2k}{k}^2$. If $k < \frac{p}{2}$, then $m-k \geq p-k > k$ and so $\binom{k}{m-k} = 0$. Thus, from the above and Lemma 2.2 we deduce

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &\equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} \\ &= \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{m=k}^{p-1} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} x^{m-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=0}^{p-1-k} \binom{2r}{r} \binom{4r}{2r} x^r \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \left(\sum_{r=0}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r - \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \right) \\ &= \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \pmod{p^2}. \end{aligned}$$

Now suppose $0 \leq k \leq p-1$ and $p-k \leq r \leq p-1$. If $k \geq \frac{3p}{4}$, then $p^2 \nmid (2k)!$, $p^3 \mid (4k)!$ and so $\binom{2k}{k} \binom{4k}{2k} = \frac{(4k)!}{(2k)!k!^2} \equiv 0 \pmod{p^2}$. If $k < \frac{p}{4}$, then $r \geq p-k \geq \frac{3p}{4}$ and so $\binom{2r}{r} \binom{4r}{2r} = \frac{(4r)!}{(2r)!r!^2} \equiv 0 \pmod{p^2}$. If $\frac{p}{4} < k < \frac{p}{2}$, then $r \geq p-k > \frac{p}{2}$, $p \nmid (2k)!$, $p \mid (4k)!$,

$p \mid \binom{2r}{r}$ and $\binom{2k}{k} \binom{4k}{2k} = \frac{(4k)!}{(2k)!k!^2} \equiv 0 \pmod{p}$. If $\frac{p}{2} < k < \frac{3p}{4}$, then $r \geq p - k > \frac{p}{4}$, $p \mid \binom{2k}{k}$ and $\binom{2r}{r} \binom{4r}{2r} = \frac{(4r)!}{(2r)!r!^2} \equiv 0 \pmod{p}$. Hence we always have $\binom{2k}{k} \binom{4k}{2k} \binom{2r}{r} \binom{4r}{2r} \equiv 0 \pmod{p^2}$ and so

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

Corollary 2.3. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1 - \sqrt{1 - 256/m}}{128} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1 - \sqrt{1 - 256/m}}{128}$ in Theorem 2.4 we deduce the result.

Corollary 2.4. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p} \quad \text{implies} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. For $\frac{p}{4} < k < p$ we see that $\binom{2k}{k}^2 \binom{4k}{2k} = \frac{(4k)!}{k!^4} \equiv 0 \pmod{p}$. Suppose $\sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p}$. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p}.$$

Using Corollary 2.3 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1 - \sqrt{1 - 256/m}}{128} \right)^k \equiv 0 \pmod{p}$$

and so the result follows from Corollary 2.3.

Theorem 2.5. *Let $p \equiv 1, 3 \pmod{8}$ be a prime and $p = c^2 + 2d^2$ with $c, d \in \mathbb{Z}$ and $c \equiv 1 \pmod{4}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \left(2c - \frac{p}{2c} \right) \pmod{p^2}.$$

Proof. From the proof of Theorem 2.4 we know that $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $\frac{p}{4} < k < p$. By [S2, Theorem 2.1] we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c \pmod{p}.$$

Set $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} = (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp$. Then

$$\left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 = ((-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}.$$

Taking $m = 256$ in Corollary 2.3 we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \pmod{p^2}.$$

Thus, by (1.2) and the above,

$$4c^2 - 2p \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}$$

and hence $q \equiv -(-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \frac{1}{2c} \pmod{p}$. So the theorem is proved.

Remark 2.2 Theorem 2.5 is a conjecture of Zhi-Wei Sun ([Su1, Conjecture A49]). In [Su3], Zhi-Wei Sun showed that $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2}$ for primes $p \equiv 5, 7 \pmod{8}$.

Theorem 2.6 ([S1, Conjecture 2.1]). *Let $p > 3$ be a prime of the form $4k + 3$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{648^k} \equiv 0 \pmod{p^2}.$$

Proof. Since $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $p > k > \frac{p}{4}$, from [S2, Theorem 2.4] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k} \binom{4k}{2k}}{72^k} \equiv 0 \pmod{p}.$$

Thus, taking $x = \frac{1}{72}$ in Theorem 2.4 and applying the above we obtain the result.

Theorem 2.7 ([S1, Conjecture 2.2]). *Let p be a prime of the form $6k + 5$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-144)^k} \equiv 0 \pmod{p^2}.$$

Proof. Since $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $p > k > \frac{p}{4}$, from [S2, Theorem 2.5] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k} \binom{4k}{2k}}{48^k} \equiv 0 \pmod{p}.$$

Thus, taking $x = \frac{1}{48}$ in Theorem 2.4 and applying the above we obtain the result.

Theorem 2.8 ([S1, Conjecture 2.3]). *Let $p > 3$ be a prime such that $p \equiv 3, 5, 6 \pmod{7}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-3969)^k} \equiv 0 \pmod{p^2}.$$

Proof. Since $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $p > k > \frac{p}{4}$, from [S2, Theorem 2.6] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv \sum_{k=0}^{\lfloor p/4 \rfloor} \frac{\binom{2k}{k} \binom{4k}{2k}}{63^k} \equiv 0 \pmod{p}.$$

Thus, taking $x = \frac{1}{63}$ in Theorem 2.4 and applying the above we obtain the result.

3. Congruences for $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \pmod{p^2}$.

Lemma 3.1. *Let m be a nonnegative integer. Then*

$$\sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{m-k} (-432)^{m-k} = \sum_{k=0}^m \binom{3k}{k} \binom{6k}{3k} \binom{3(m-k)}{m-k} \binom{6(m-k)}{3(m-k)}.$$

We prove the lemma by using WZ method and Mathematica. Clearly the result is true for $m = 0, 1$. Since both sides satisfy the same recurrence relation

$$20736(m+1)(3m+1)(3m+5)S(m) - 24(2m+3)(18m^2+54m+41)S(m+1) + (m+2)^3 S(m+2) = 0,$$

we see that Lemma 3.1 is true. The proof certificate for the left hand side is

$$-\frac{186624k^2(m+2)(m-2k)(m-2k+1)}{(m-k+1)(m-k+2)},$$

and the proof certificate for the right hand side is

$$\frac{144k^2(6m-6k+1)(6m-6k+5)(36m^2-36mk+129m-62k+114)}{(m-k+1)^2(m-k+2)^2}.$$

For given prime p and integer n , if $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$, we say that $p^\alpha \parallel n$.

Lemma 3.2. *Let p be an odd prime, $k, r \in \{0, 1, \dots, p-1\}$ and $k+r \geq p$. Then*

$$\binom{3k}{k} \binom{6k}{3k} \binom{3r}{r} \binom{6r}{3r} \equiv 0 \pmod{p^2}.$$

Proof. If $k > \frac{5p}{6}$, then $p^5 \mid (6k)!$, $p \parallel (2k)!$, $p^2 \parallel (3k)!$ and so $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p^2}$. If $\frac{2p}{3} \leq k < \frac{5p}{6}$, then $2p \leq 3k < 3p$, $4p \leq 6k < 5p$, $p^4 \parallel (6k)!$, $p^2 \parallel (3k)!$, $p \parallel (2k)!$ and so $\binom{3k}{k} \binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$. If $\frac{p}{2} < k < \frac{2p}{3}$, then $p < 3k < 2p$,

$3p < 6k < 4p$, $p^3 \mid (6k)!$, $p \parallel (2k)!$, $p \parallel (3k)!$ and so $\binom{3k}{k}\binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$. If $\frac{p}{3} \leq k < \frac{p}{2}$, then $2k < p$, $p \leq 3k < 2p$, $6k \geq 2p$, $p^2 \mid (6k)!$, $p \nmid (2k)!$, $p \parallel (3k)!$ and so $\binom{3k}{k}\binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$. If $\frac{p}{6} < k < \frac{p}{3}$, then $3k < p$, $6k > p$ and so $\binom{3k}{k}\binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p}$.

From the above we see that $p \mid \binom{3k}{k}\binom{6k}{3k}$ for $k > \frac{p}{6}$. Therefore, if $k > \frac{p}{6}$ and $r > \frac{p}{6}$, then $\binom{3k}{k}\binom{6k}{3k}\binom{3r}{r}\binom{6r}{3r} \equiv 0 \pmod{p^2}$. If $r < \frac{p}{6}$, then $k \geq p - r > \frac{5p}{6}$ and so $p^2 \mid \binom{3k}{k}\binom{6k}{3k}$ by the above. If $k < \frac{p}{6}$, then $r \geq p - k > \frac{5p}{6}$ and so $p^2 \mid \binom{3r}{r}\binom{6r}{3r}$ by the above.

Now putting all the above together we prove the lemma.

Theorem 3.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1 - 432x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1 - 432x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=0}^k \binom{k}{r} (-432x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{m-k} (-432)^{m-k}. \end{aligned}$$

Suppose $p \leq m \leq 2p - 2$ and $0 \leq k \leq p - 1$. If $k \geq \frac{2p}{3}$, then $2p \leq 3k < 3p$, $6k \geq 4p$, $p^3 \nmid (3k)!$, $p^4 \mid (6k)!$ and so $\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k} = \frac{(6k)!}{(3k)!k!^3} \equiv 0 \pmod{p^2}$. If $\frac{p}{2} < k < \frac{2p}{3}$, then $3k < 2p$, $6k > 3p$, $p^2 \nmid (3k)!$ and $p^3 \mid (6k)!$ and so $\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k} = \frac{(6k)!}{(3k)!k!^3} \equiv 0 \pmod{p^2}$. If $k < \frac{p}{2}$, then $m - k \geq p - k > k$ and so $\binom{k}{m-k} = 0$. Thus, from the above and Lemma 3.1

we deduce

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} (x(1-432x))^k \\
& \equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \binom{k}{m-k} (-432)^{m-k} \\
& = \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{3k}{k} \binom{6k}{3k} \binom{3(m-k)}{m-k} \binom{6(m-k)}{3(m-k)} \\
& = \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{m=k}^{p-1} \binom{3(m-k)}{m-k} \binom{6(m-k)}{3(m-k)} x^{m-k} \\
& = \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=0}^{p-1-k} \binom{3r}{r} \binom{6r}{3r} x^r \\
& = \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \left(\sum_{r=0}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r - \sum_{r=p-k}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r \right) \\
& = \left(\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=p-k}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r \pmod{p^2}.
\end{aligned}$$

By Lemma 3.2, we have $p^2 \mid \binom{3k}{k} \binom{6k}{3k} \binom{3r}{r} \binom{6r}{3r}$ for $0 \leq k \leq p-1$ and $p-k \leq r \leq p-1$. Thus

$$\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} x^k \sum_{r=p-k}^{p-1} \binom{3r}{r} \binom{6r}{3r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

Corollary 3.1. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} \left(\frac{1 - \sqrt{1 - 1728/m}}{864} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1 - \sqrt{1 - 1728/m}}{864}$ in Theorem 3.1 we deduce the result.

Corollary 3.2. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{[p/6]} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv 0 \pmod{p} \quad \text{implies} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. From the proof of Lemma 3.2 we know that $p \mid \binom{3k}{k} \binom{6k}{3k}$ for $p > k > \frac{p}{6}$. Suppose $\sum_{k=0}^{[p/6]} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv 0 \pmod{p}$. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv \sum_{k=0}^{[p/6]} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{m^k} \equiv 0 \pmod{p}.$$

Using Corollary 3.1 we see that

$$\sum_{k=0}^{p-1} \binom{3k}{k} \binom{6k}{3k} \left(\frac{1 - \sqrt{1 - 1728/m}}{864} \right)^k \equiv 0 \pmod{p}$$

and so the result follows.

Theorem 3.2. *Let $p \equiv 1 \pmod{4}$ be a prime and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $4 \mid a - 1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{864^k} \equiv \begin{cases} 2a - \frac{p}{2a} \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -2a + \frac{p}{2a} \pmod{p^2} & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ 2b - \frac{p}{2b} \pmod{p^2} & \text{if } p \equiv 5 \pmod{12} \text{ and } 3 \mid a - b. \end{cases}$$

Proof. By the proof of Lemma 3.2, we have $p \mid \binom{3k}{k} \binom{6k}{3k}$ for $p > k > \frac{p}{6}$. Set

$$r = \begin{cases} a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ b & \text{if } p \equiv 5 \pmod{12} \text{ and } 3 \mid a - b. \end{cases}$$

Using [S3, Theorem 2.1] we have

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \equiv \sum_{k=0}^{[p/6]} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \equiv 2r \pmod{p}.$$

Set $\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} = 2r + qp$. Then

$$\left(\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \right)^2 = (2r + qp)^2 \equiv 4r^2 + 4rqp \pmod{p^2}.$$

Taking $m = 1728$ in Corollary 3.1 we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{1728^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \right)^2 \pmod{p^2}.$$

Thus, by (1.3) and the above,

$$\binom{p}{3} (4a^2 - 2p) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{1728^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \right)^2 \equiv 4r^2 + 4rqp \pmod{p^2}$$

and hence $q \equiv -\frac{1}{2r} \pmod{p}$. So the theorem is proved.

Remark 3.1 Theorem 3.2 is a conjecture of Zhi-Wei Sun ([Su1, Conjecture A44]). In [Su3], Zhi-Wei Sun showed that $\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \equiv 0 \pmod{p^2}$ for any prime $p = 4n + 3 > 3$.

Theorem 3.3 ([S1, Conjectures 2.8 and 2.9]). *Let $p > 7$ be a prime such that $p \equiv 3, 5, 6 \pmod{7}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-15)^{3k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{255^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorems 3.9 and 3.10] and Corollary 3.2.

Theorem 3.4 ([Su1, Conjecture A26]). *Let p be an odd prime with $p \equiv 2, 6, 7, 8, 10 \pmod{11}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-32)^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.4] and Corollary 3.2.

Theorem 3.5 ([Su1, Conjecture A9]). *Let $p > 3$ be a prime with $(\frac{p}{19}) = -1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-96)^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.5] and Corollary 3.2.

Theorem 3.6 ([Su1, Conjecture A10]). *Let $p > 5$ be a prime with $(\frac{p}{43}) = -1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-960)^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.6] and Corollary 3.2.

Theorem 3.7 ([Su1, Conjecture A11]). *Let $p > 5$ be a prime with $p \neq 11$ and $(\frac{p}{67}) = -1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-5280)^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.7] and Corollary 3.2.

Theorem 3.8 ([Su1, Conjecture A12]). *Let $p \neq 2, 3, 5, 23, 29$ be a prime with $(\frac{p}{163}) = -1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-640320)^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.8] and Corollary 3.2.

Theorem 3.9 ([S1, Conjecture 2.4]). *Let $p \neq 3, 11$ be a prime such that $p \equiv 3 \pmod{4}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{66^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.11] and Corollary 3.2.

Theorem 3.10 ([S1, Conjecture 2.5]). *Let $p > 5$ be a prime such that $p \equiv 5, 7 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{20^{3k}} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.12] and Corollary 3.2.

Theorem 3.11 ([S1, Conjecture 2.6]). *Let $p > 5$ be a prime such that $p \equiv 2 \pmod{3}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{54000^k} \equiv 0 \pmod{p^2}.$$

Proof. This is immediate from [S3, Theorem 3.13] and Corollary 3.2.

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